

QUARK-ANTIQUARK BETHE-SALPETER FORMALISM, SPECTRUM AND REGGE TRAJECTORIES

G. M. Prosperi

*Dipartimento di Fisica, Università di Milano,
INFN, sezione di Milano,
via Celoria 16, I20133 Milano, Italy*

Sept 4, 1998

Abstract

Starting from a path integral representation of appropriate 4-point and 2-point gauge invariant Green functions and from the "Modified Area Law" model, a $q\bar{q}$ Bethe-Salpeter like equation and a related Schwinger-Dyson equation can be obtained. From such equations an effective relativistic Hamiltonian can be derived by standard methods and then applied to the determination of the meson spectrum. The entire known heavy-heavy and heavy-light spectra and the lowest light-light Regge trajectories are rather well reproduced in terms of four parameters alone, the light quark masses being fixed a priori on typical current values.

1 Introduction

In this paper I want to review a Bethe Salpeter formalism in QCD, which has been developed recently in Milano. Such formalism, if not completely derived from first principles, rests, however, only on some non controversial assumptions on the Wilson loop correlator [1]. It generalizes a method introduced previously for the case of the heavy quark potential [2] and takes advantage of appropriate Feynmann-Schwinger like path integral representations for the QCD Green functions. Its three dimensional reduction has been recently applied to the light-light and heavy-light quark-antiquark spectrum with very encouraging results [4]. Beside the heavy quarkonia, the lowest Regge trajectories for the triplet $u\bar{u}$, $u\bar{s}$, $s\bar{s}$ states are very well reproduced in slope and intercepts and the known spin averaged light-heavy spectrum is obtained up to a mean deviation of about 10 MeV. Notice that it is

very difficult to obtain a similar result in the frame of a single model. In fact in our case the strong coupling constant α_s , the string tension σ and the heavy quark masses m_b and m_c were already completely determined by the potential fits (apart the possibility of a very small rearrangement), while the light quark masses have been fixed a priori on typical current values ($m_u = m_d = 10$ MeV, $m_s = 200$ MeV). In particular the heavy-light sector is completely parameter free.

The basic objects from which we start are the ordinary gauge invariant 4-point and 2-point Green functions $G^{\text{gi}}(x_1, x_2, y_1, y_2)$ and $G^{\text{gi}}(x - y)$, to which we refer as the “first order” functions. To these first order functions certain “second order” ones $H^{\text{gi}}(x_1, x_2, y_1, y_2)$ and $H^{\text{gi}}(x - y)$ can be related. It is for the second order functions that the mentioned path integral representations can be constructed.

The important aspect of the above representations is that the gauge field occurs simply through Wilson correlators

$$W[\Gamma] = \frac{1}{3} \langle \text{TrP} \exp \{ ig \oint_{\Gamma} dx^{\mu} A_{\mu} \} \rangle . \quad (1)$$

associated to loops Γ made by quark or antiquark world lines and “Schwinger strings”. In principle these correlators should determine the whole dynamics. Unfortunately, due to confinement and the consequent failure of a purely perturbative approach, a consistent analytic evaluation of W from the Lagrangian alone is not possible today. However combining incomplete theoretical arguments and lattice simulation information various reasonable models can be attempted.

The most naive but at the same time less arbitrary assumption consists in writing $i \ln W$ as the sum of its perturbative expression and an area term (modified area law (MAL) model)

$$i \ln W = i(\ln W)_{\text{pert}} + \sigma S_{\text{min}} , \quad (2)$$

where the first quantity is supposed to give correctly the short range limit the second the long range one.

Eq. (2) is our starting point. In principle any more sophisticated model could be used in the context, at the condition that it preserves certain general properties of functional derivability of the definition (1). In practice not even (2) can be treated exactly. Actually we shall replace the minimal surface S_{min} by what can be called its “equal time straight line approximation”, which we shall explain later.

In connection with (2) it is convenient to consider a third type of Green functions $H(x_1, x_2, y_1, y_2)$ and $H(x - y)$ which are obtained from $H^{\text{gi}}(x_1, x_2, y_1, y_2)$ and $H^{\text{gi}}(x - y)$ by omitting in their path integral representation contributions related to the Schwinger strings. In the limit of vanishing $x_1 - x_2$, $y_1 - y_2$ or $x - y$ such new quantities coincide with the original ones and are completely equivalent for what concerns the determination of bound states, condensates, chiral symmetry breaking, etc.

For $H(x_1, x_2, y_1, y_2)$ and $H(x - y)$, an inhomogeneous Bethe-Salpeter equation and a Dyson-Schwinger equation, respectively, can be derived in the configuration space, with kernel obtained as an expansion in α_s and σ . Such equations can be

rewritten in a more conventional form in the momentum space and as such applied e. g. to the spectrum of the mesons. To this aim in principle one should solve the DS equation first and use the resulting propagator in the BS equation. In practice a direct treatment of their full four dimensional expressions seems to be a very difficult task. However, if one neglects the pseudo-scalar mesons, a three dimensional reduction of the BS-equation seems to be appropriate. This can be obtained by standard methods in the form of an eigenvalue equation for an effective squared mass operator or simply a relativistic Hamiltonian [1, 2].

For the case of the pseudo-scalar mesons a complete four-dimensional treatment would be imperative. In fact the use of the free quark propagator implied in the three dimensional reduction can not be suitable in this case, for the strict interplay existing between zero mass Bethe-Salpeter wave function and the propagator in the chiral limit; we shall limit to few comments.

It should be mentioned that our formalism is strictly related from different point of view to the works of ref. [5].

The plan of the paper is the following one. We discuss the gauge invariant Green functions and their path integral representations in sect. 2; modified area law model and straight line approximation in sect. 3; Bethe-Salpeter, Dyson-Schwinger equations and the problem of chiral symmetry in sect. 4. In sect. 5 we discuss the three dimensional reduction of the BS-equation and its application to the determination of the spectrum and the Regge trajectories.

2 Green functions and Feynman-Schwinger representations

The quark-antiquark and the single quark gauge invariant Green functions are defined as

$$\begin{aligned} G^{\text{gi}}(x_1, x_2, y_1, y_2) &= \frac{1}{3} \langle 0 | T \psi_2^c(x_2) U(x_2, x_1) \psi_1(x_1) \bar{\psi}_1(y_1) U(y_1, y_2) \bar{\psi}_2^c(y_2) | 0 \rangle = \\ &= \frac{1}{3} \text{Tr}_C \langle U(x_2, x_1) S_1(x_1, y_1; A) U(y_1, y_2) \tilde{S}_2(y_2, x_2; -\tilde{A}) \rangle \end{aligned} \quad (3)$$

(plus an annihilation term in the equal flavor case) and

$$G^{\text{gi}}(x - y) = \langle 0 | T U(y, x) \psi(x) \bar{\psi}(y) | 0 \rangle = i \text{Tr}_C \langle U(y, x) S(x, y; A) \rangle, \quad (4)$$

where ψ^c denotes the charge-conjugate fields, the tilde and Tr_C the transposition and the trace respectively over the color indices alone and U the path-ordered gauge string (Schwinger string)

$$U(b, a) = \text{P exp} \left\{ ig \int_a^b dx^\mu A_\mu(x) \right\}. \quad (5)$$

Notice that the integration in (5) is along an arbitrary line joining a to b (which usually we shall not specify explicitly), S , S_1 and S_2 are the quark propagators in the external gauge field A^μ and the angle brackets denote average on the gauge variable alone (weighted in principle with the determinant $M_f(A)$ resulting from the explicit integration of the fermionic fields).

The propagator S is supposed to be defined by the equation (we shall suppress indices specifying the quarks, as a rule, when dealing with single quark quantities)

$$(i\gamma^\mu D_\mu - m)S(x, y; A) = \delta^4(x - y) \quad (6)$$

and the appropriate boundary conditions. This can be rewritten as [1]

$$S(x, y; A) = (i\gamma^\nu D_\nu + m)\Delta^\sigma(x, y; A), \quad (7)$$

in terms of a “second order” propagator defined in turn by the equation

$$(D_\mu D^\mu + m^2 - \frac{1}{2}g\sigma^{\mu\nu}F_{\mu\nu})\Delta^\sigma(x, y; A) = -\delta^4(x - y), \quad (8)$$

with $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$.

After replacing (7) in (3) and (4), using an appropriate derivative it is possible to take the differential operator out of the angle brackets and write ¹

$$G^{\text{gi}}(x_1, x_2; y_1, y_2) = -(i\gamma_1^\mu \bar{\partial}_{1\mu} + m_1)(i\gamma_2^\nu \bar{\partial}_{2\nu} + m_2)H^{\text{gi}}(x_1, x_2; y_1, y_2), \quad (9)$$

$$G^{\text{gi}}(x - y) = (i\gamma_1^\mu \bar{\partial}_\mu + m)H^{\text{gi}}(x - y), \quad (10)$$

with

$$H^{\text{gi}}(x_1, x_2; y_1, y_2) = -\frac{1}{3}\text{Tr}_C \langle U(x_2, x_1)\Delta_1^\sigma(x_1, y_1; A)U(y_1, y_2)\tilde{\Delta}_2^\sigma(x_2, y_2; -\tilde{A}) \rangle, \quad (11)$$

$$H^{\text{gi}}(x - y) = i\text{Tr}_C \langle U(y, x)\Delta^\sigma(x, y; A) \rangle. \quad (12)$$

For the second order propagator we have the Feynman-Schwinger representation

$$\Delta^\sigma(x, y; A) = -\frac{i}{2} \int_0^\infty ds \int_y^x \mathcal{D}z \exp[-i \int_0^s d\tau \frac{1}{2}(m^2 + \dot{z}^2)] \mathcal{S}_0^s \text{P} \exp[ig \int_0^s d\tau \dot{z}^\mu A_\mu(z)], \quad (13)$$

with

$$\mathcal{S}_0^s = \text{T} \exp \left[-\frac{1}{4} \int_0^s d\tau \sigma^{\mu\nu} \frac{\delta}{\delta S^{\mu\nu}(z)} \right], \quad (14)$$

¹Given a functional $\Phi[\gamma_{ab}]$ of the curve γ_{ab} with ends a and b , let us assume that the variation of Φ consequent to an infinitesimal modification of the curve $\gamma \rightarrow \gamma + \delta\gamma$ can be expressed as the sum of various terms proportional respectively to δa , to δb and to the single elements $\delta S^{\rho\sigma}(x)$ of the surface swept by the curve. Then, the derivatives $\bar{\partial}_{a\rho}$, $\bar{\partial}_{b\rho}$ and $\delta/\delta S^{\rho\sigma}(x)$ are defined by the equation $\delta\Phi = \delta a^\rho \bar{\partial}_{a\rho}\Phi + \delta b^\rho \bar{\partial}_{b\rho}\Phi + \frac{1}{2} \int_\gamma \delta S^{\rho\sigma}(x) \delta\Phi/\delta S^{\rho\sigma}(x)$. For a Schwinger string we have $\delta U(b, a) = \delta b^\rho ig A_\rho(b)U(b, a) - \delta a^\rho U(b, a)ig A_\rho(a) + \frac{ig}{2} \int_a^b \delta S^{\rho\sigma}(z) \text{P}(-F_{\rho\sigma}(z)U(b, a)$ and so $\bar{\partial}_{a\rho}U = -igUA_\rho(a)$, $\bar{\partial}_{b\rho}U = igA_\rho(b)U$ and $\frac{\delta}{\delta S^{\rho\sigma}(z)}U = \text{P}[-igF_{\rho\sigma}(z)U]$.

T and P being the ordering prescriptions along the path acting on the spin and on the color matrices respectively and $\delta S^{\mu\nu} = dz^\mu \delta z^\nu - dz^\nu \delta z^\mu$ (the functional derivative being defined through an arbitrary deformation, $z \rightarrow z + \delta z$, of the line connecting a to b , see footnote) [1]. Replacing (13) in (11) and (12) we obtain

$$\begin{aligned} H^{\mathbf{gi}}(x_1, x_2; y_1, y_2) &= \left(\frac{1}{2}\right)^2 \int_0^\infty ds_1 \int_0^\infty ds_2 \int_{y_1}^{x_1} \mathcal{D}z_1 \int_{y_2}^{x_2} \mathcal{D}z_2 \\ &\exp \left\{ -\frac{i}{2} \int_0^{s_1} d\tau_1 (m_1^2 + \dot{z}_1^2) - \frac{i}{2} \int_0^{s_2} d\tau_2 (m_2^2 + \dot{z}_2^2) \right\} \\ &\mathcal{S}_0^{s_1} \mathcal{S}_0^{s_2} \frac{1}{3} \langle \text{Tr P exp} \{ ig \oint_{\Gamma_{\bar{q}q}} dz^\mu A_\mu(z) \} \rangle, \end{aligned} \quad (15)$$

$$\begin{aligned} H^{\mathbf{gi}}(x - y) &= \frac{1}{2} \int_0^\infty ds \int_y^x \mathcal{D}z \exp \left\{ -\frac{i}{2} \int_0^s d\tau (m^2 + \dot{z}^2) \right. \\ &\left. \mathcal{S}_0^s \langle \text{Tr P exp} \{ ig \oint_{\Gamma_q} dz^\mu A_\mu(z) \} \rangle \right\}. \end{aligned} \quad (16)$$

Here, the loop $\Gamma_{\bar{q}q}$ occurring in the 4-points function is made by the quark world line γ_1 , the antiquark world line γ_2 followed in the reverse direction, and the two strings $x_1 x_2$ and $y_2 y_1$; the loop Γ_q occurring in the 2-points function is made simply by the quark trajectory γ connecting y to x and the string yx .

3 Wilson loop correlators

We now apply Eq. (2) to evaluate the Wilson correlators for $\Gamma_{\bar{q}q}$ and Γ_q .

At the lowest order the perturbative term can be written for any loop Γ

$$i(\ln W[\Gamma])_{\text{pert}} = -\frac{2}{3} g^2 \oint dz^\mu \oint dz^{\nu'} D_{\mu\nu}(z - z') \quad (17)$$

$D_{\mu\nu}(z - z')$ being the free gauge propagator. If we neglect the contribution coming from propagators connecting a point on a world-line to a point on a string or two point on the strings, we can write for $\Gamma_{\bar{q}\bar{q}}$

$$\begin{aligned} i(\ln W[\Gamma_{\bar{q}\bar{q}}])_{\text{pert}} &= \frac{4}{3} g^2 \int_0^{s_1} d\tau_1 \int_0^{s_2} d\tau_2 D_{\mu\nu}(z_1 - z_2) \dot{z}_1^\mu \dot{z}_2^\nu - \\ &\quad - \frac{4}{3} g^2 \sum_{j=1}^2 \int_0^{s_j} d\tau_j \int_0^{\tau_j} d\tau'_j D_{\mu\nu}(z_j - z'_j) \dot{z}_j^\mu \dot{z}_j^{\nu'} \end{aligned} \quad (18)$$

and for Γ_q

$$i(\ln W[\Gamma_q])_{\text{pert}} = -\frac{4}{3} g^2 \int_0^s d\tau \int_0^\tau d\tau' D_{\mu\nu}(z - z') \dot{z}^\mu \dot{z}'^\nu \quad (19)$$

Let us further consider for the moment the case in which $\Gamma_{q\bar{q}}$ lies on a plane. Then S_{\min} coincides simply with the portion of plane delimited by $\Gamma_{q\bar{q}}$ and it can be written, in a four dimensional language, [1]

$$\begin{aligned}
S_{\min} = & \int_0^{s_1} d\tau_1 \int_0^{s_2} d\tau_2 \delta(z_{10} - z_{20}) |\mathbf{z}_1 - \mathbf{z}_2| \epsilon(\dot{z}_{10}) \epsilon(\dot{z}_{20}) \\
& \int_0^1 d\lambda \left\{ \dot{z}_{10}^2 \dot{z}_{20}^2 - (\lambda \dot{\mathbf{z}}_{1T} \dot{z}_{20} + (1 - \lambda) \dot{\mathbf{z}}_{2T} \dot{z}_{10})^2 \right\}^{\frac{1}{2}} - \\
& - \sum_{j=1}^2 \int_0^{s_j} d\tau_j \int_0^{\tau_j} d\tau'_j \delta(z_{j0} - z'_{j0}) |\mathbf{z}_j - \mathbf{z}'_j| \epsilon(\dot{z}_{j0}) \epsilon(\dot{z}'_{j0}) \\
& \int_0^j d\lambda \left\{ \dot{z}_{j0}^2 \dot{z}'_{j0}{}^2 - (\lambda \dot{\mathbf{z}}_{1T} \dot{z}'_{j0} + (1 - \lambda) \dot{\mathbf{z}}'_{jT} \dot{z}_{j0})^2 \right\}^{\frac{1}{2}}, \quad (20)
\end{aligned}$$

where we have used z'_j for $z_j(\tau'_j)$ and $\epsilon(t)$ denotes the sign function. Eq. (20) corresponds to span the surface by a straight line joining two points with the same time coordinate on the quark and the antiquark world lines respectively. The sign factors and the second term are necessary to reconstruct the surface as the algebraic sum of various pieces when the world-lines go backward in time.

The “straight line equal time approximation” consists in assuming (20) even if $\Gamma_{q\bar{q}}$ does not stay on a plane. Notice that (20) gives always S_{\min} correctly up to the order $(\dot{\mathbf{z}}/\dot{z}_0)^2$ in a semi-relativistic expansion. Furthermore (20) is exact for particular geometries. This is the case e. g. for γ_1 and γ_2 making a regular double helix (with axis parallel to the time axis) corresponding to a pure rotational motion of the quark and the antiquark around a fixed point. Notice also that the approximation depends in general on the reference frame. Keeping in mind the helix example, however, we shall assume its validity in the center of mass frame.

In a similar way, in the case of Γ_q we can set for $x^0 = y^0$ we can set

$$\begin{aligned}
S_{\min} = & - \int_0^s d\tau \int_0^\tau d\tau' \delta(z_0 - z'_0) |\mathbf{z} - \mathbf{z}'| \epsilon(\dot{z}_0) \epsilon(\dot{z}'_0) \int_0^1 d\lambda \left\{ \dot{z}_0^2 \dot{z}'_0{}^2 - \right. \\
& \left. - (\lambda \dot{\mathbf{z}}_T \dot{z}'_0 + (1 - \lambda) \dot{\mathbf{z}}'_T \dot{z}_0)^2 \right\}^{\frac{1}{2}}. \quad (21)
\end{aligned}$$

It is clear that, to include consistently the cases $x_1^0 \neq x_2^0$, $y_1^0 \neq y_2^0$, $x^0 \neq y^0$, the line integrals in (20) and (21) should be extended in an obvious way to the Schwinger strings. However, we find convenient to consider two new functions that are obtained replacing (18) and (20) in (15) and (19) and (21) in (16) so as they stand. We obtain in this way the following equations

$$\begin{aligned}
H(x_1, x_2; y_1, y_2) = & \left(\frac{1}{2}\right)^2 \int_0^\infty ds_1 \int_0^\infty ds_2 \int_{y_1}^{x_1} \mathcal{D}z_1 \int_{y_2}^{x_2} \mathcal{D}z_2 \\
& \exp \left\{ -\frac{i}{2} \sum_{j=1}^2 \int_0^{s_j} d\tau_j (m_j^2 + \dot{z}_j^2) \right\} \mathcal{S}_0^{s_1} \mathcal{S}_0^{s_2} \exp \left\{ i \sum_{j=1}^2 \int_0^{s_j} d\tau_j \int_0^{\tau_j} d\tau'_j E(z_j - z'_j; \dot{z}_j, \dot{z}'_j) \right. \\
& \left. - i \int_0^{s_1} d\tau_1 \int_0^{s_2} d\tau_2 E(z_1 - z_2; \dot{z}_1, \dot{z}_2) \right\} \quad (22)
\end{aligned}$$

and

$$H(x-y) = \frac{1}{2} \int_0^\infty ds \int_y^x \mathcal{D}z \exp \left\{ -\frac{i}{2} \int_0^s d\tau (m^2 + \dot{z}^2) \right\} \mathcal{S}_0^s \exp \left\{ i \int_0^s \int_0^\tau E(z-z'; \dot{z}, \dot{z}') \right\}, \quad (23)$$

where we have set

$$E(\zeta; p, p') = E_{\text{pert}}(\zeta; p, p') + E_{\text{conf}}(\zeta; p, p') \quad (24)$$

with

$$\begin{cases} E_{\text{pert}} &= 4\pi \frac{4}{3} \alpha_s D_{\mu\nu}(\zeta) p^\mu p'^\nu \\ E_{\text{conf}} &= \delta(\zeta_0) |\zeta| \epsilon(p_0) \epsilon(p'_0) \int_0^1 d\lambda \{ p_0^2 p_0'^2 - [\lambda p'_0 \mathbf{p}_T + (1-\lambda) p_0 \mathbf{p}'_T]^2 \}^{\frac{1}{2}} \end{cases} \quad (25)$$

Obviously for arbitrary arguments the quantities $H(x_1, x_2; y_1, y_2)$ and $H(x-y)$ as defined by (22)-(25) can differ very significantly from the original $H^{\text{gi}}(x_1, x_2; y_1, y_2)$ and $H^{\text{gi}}(x-y)$. However, as we mentioned, the two couples coincide in the limits $x_2 \rightarrow x_1$, $y_2 \rightarrow y_1$, and $y \rightarrow x$ and they are completely equivalent for what concerns bound state problems and condensate determination.

4 Bethe-Salpeter and Dyson-Schwinger equations

From eq.s (22) and (23), by various manipulation and using an appropriate iterative procedure, a Bethe-Salpeter equation for the function $H(x_1, x_2; y_1, y_2)$ and a Dyson-Schwinger equation for $H(x-y)$ can be derived in the form [1]

$$\begin{aligned} H(x_1, x_2; y_1, y_2) &= H_1(x_1 - y_1) H_2(x_2 - y_2) - \\ &- i \int d^4 \xi_1 d^4 \xi_2 d^4 \eta_1 d^4 \eta_2 H_1(x_1 - \xi_1) H_2(x_2 - \xi_2) \\ &\times I_{ab}(\xi_1, \xi_2; \eta_1, \eta_2) \sigma_1^a \sigma_2^b H(\eta_1, \eta_2; y_1, y_2), \end{aligned} \quad (26)$$

$$\begin{aligned} H(x-y) &= H_0(x-y) + i \int d^4 \xi d^4 \eta d^4 \xi' d^4 \eta' H_0(x-\xi) \\ &\times I_{ab}(\xi, \xi'; \eta, \eta') \sigma^a H(\eta - \eta') \sigma^b H(\xi' - y), \end{aligned} \quad (27)$$

where we have set $a, b = 0$, $\mu\nu$, with $\sigma^0 = 1$, and H_1 and H_2 denote the quark and the antiquark H-propagators respectively.

If we pass to the momentum representation, the corresponding homogeneous BS-equation becomes in a 4×4 matrix representation

$$\begin{aligned} \Phi_P(k) &= -i \int \frac{d^4 u}{(2\pi)^4} \hat{I}_{ab}(k-u, \frac{1}{2}P + \frac{k+u}{2}, \frac{1}{2}P - \frac{k+u}{2}) \\ &\hat{H}_1(\frac{1}{2}P + k) \sigma^a \Phi_P(u) \sigma^b \hat{H}_2(-\frac{1}{2}P + k), \end{aligned} \quad (28)$$

where $\Phi_P(k)$ denotes an appropriate wave function and the center of mass frame has to be understood; i.e. $P = (m_B, \mathbf{0})$.

Similarly, in terms of the irreducible self-energy, defined by $\hat{H}(k) = \hat{H}_0(k) + i\hat{H}_0(k)\hat{\Gamma}(k)\hat{H}(k)$, the DS-equation can be written also

$$\hat{\Gamma}(k) = \int \frac{d^4l}{(2\pi)^4} \hat{I}_{ab}\left(k-l; \frac{k+l}{2}, \frac{k+l}{2}\right) \sigma^a \hat{H}(l) \sigma^b. \quad (29)$$

Notice that in principle (26) and (27) or (28) and (29) are exact equations. However the kernels I_{ab} are generated in the form of an expansion in α_s and σ . At the lowest order in both such constants, we have explicitly

$$\begin{aligned} \hat{I}_{0;0}(Q; p, p') &= 4 \int d^4\zeta e^{iQ\cdot\zeta} E(\zeta; p, p') = 16\pi \frac{4}{3} \alpha_s p^\alpha p'^\beta \hat{D}_{\alpha\beta}(Q) + \\ &\quad + 4\sigma \int d^3\zeta e^{-i\mathbf{Q}\cdot\zeta} |\zeta| \epsilon(p_0) \epsilon(p'_0) \int_0^1 d\lambda \{p_0^2 p_0'^2 - [\lambda p'_0 \mathbf{p}_T + (1-\lambda) p_0 \mathbf{p}'_T]^2\}^{\frac{1}{2}} \\ \hat{I}_{\mu\nu;0}(Q; p, p') &= 4\pi i \frac{4}{3} \alpha_s (\delta_\mu^\alpha Q_\nu - \delta_\nu^\alpha Q_\mu) p'_\beta \hat{D}_{\alpha\beta}(Q) - \\ &\quad - \sigma \int d^3\zeta e^{-i\mathbf{Q}\cdot\zeta} \epsilon(p_0) \frac{\zeta_\mu p_\nu - \zeta_\nu p_\mu}{|\zeta| \sqrt{p_0^2 - \mathbf{p}_T^2}} p'_0 \\ \hat{I}_{0;\rho\sigma}(Q; p, p') &= -4\pi i \frac{4}{3} \alpha_s p^\alpha (\delta_\rho^\beta Q_\sigma - \delta_\sigma^\beta Q_\rho) \hat{D}_{\alpha\beta}(Q) + \\ &\quad + \sigma \int d^3\zeta e^{-i\mathbf{Q}\cdot\zeta} p_0 \frac{\zeta_\rho p'_\sigma - \zeta_\sigma p'_\rho}{|\zeta| \sqrt{p_0^2 - \mathbf{p}_T^2}} \epsilon(p'_0) \\ \hat{I}_{\mu\nu;\rho\sigma}(Q; p, p') &= \pi \frac{4}{3} \alpha_s (\delta_\mu^\alpha Q_\nu - \delta_\nu^\alpha Q_\mu) (\delta_\rho^\alpha Q_\sigma - \delta_\sigma^\alpha Q_\rho) \hat{D}_{\alpha\beta}(Q) \end{aligned} \quad (30)$$

where in the second and in the third equation $\zeta_0 = 0$ has to be understood.

Setting

$$i\hat{H}^{-1}(k) = \sum_{r=0}^3 \omega_r(k) h_r(k), \quad (31)$$

with $\omega_0 = 1$, $\omega_1 = \gamma^0$, $\omega_2 = -\gamma \cdot \hat{\mathbf{k}}$, $\omega_3 = \gamma^0 \gamma \cdot \hat{\mathbf{k}}$, $\hat{\mathbf{k}} = \frac{1}{|\mathbf{k}|} \mathbf{k}$ and $h_0(k), \dots, h_3(k)$ functions of k_0 and $|\mathbf{k}|$. Eq.(29) can also be written

$$h_r(k) = \delta_{r0}(k^2 - m^2) - i \sum_{s=0}^3 \int \frac{d^4l}{(2\pi)^4} \frac{R_{rs}(k, l) h_s(l)}{h_0^2(l) - h_1^2(l) + h_2^2(l) - h_3^2(l)}, \quad (32)$$

$$R_{rs}(k, l) = \mp \frac{1}{4} \hat{I}_{ab}\left(k-l; \frac{k+l}{2}, \frac{k+l}{2}\right) \text{Tr}[\omega_r^+(k) \sigma^a \omega_s(k) \sigma^b], \quad (33)$$

where the sign $-$ applies to the $s = 0$ case, the sign $+$ to all the other cases. Notice that actually only R_{00} , R_{11} , R_{12} , R_{21} , R_{22} , R_{33} are different from zero.

In connection with the problem of the light pseudo-scalar mesons, let us now consider eq.s (28) and (32) in the chiral limit $m_1 = m_2 = 0$. As it is apparent from

(10) chiral symmetry is broken in such limit, if h_1 and h_2 do not both vanish. On the other side, if this is the case and if in addition $h_3 \rightarrow 0$, it can be checked that eq. (28) is solved for $P = 0$ by $\Phi_0(k) = \hat{H}_1(k)[\gamma^0 h_1(k) + \gamma \cdot \mathbf{k} h_2(k)]\gamma^5 \hat{H}_2(k)$, and a zero mass pseudo-scalar bound state exists, consistently with the Goldstone theorem. Notice that such a result is strictly related to the occurrence of the same kernel in the two equations and this in turn is due to the inclusion of the self-energy terms in (20), i. e. to the correct account of world lines that go backwards in time. Obviously the supposed behavior of the solution is superficially consistent with the form of (32); presently, however, we are not able to produce any proof.

5 Three-dimensional reduction and spectrum

By replacing $\hat{H}_1(k)$ and $\hat{H}_2(k)$ in the BS equation by the corresponding free propagators $\frac{i}{k^2 - m_j^2}$ and performing an instantaneous approximation on the kernels, one can obtain a three dimensional reduction of the original equation in the form of the eigenvalue equation for the relativistic Hamiltonian [1]

$$H = \sqrt{m_1^2 + \mathbf{k}^2} + \sqrt{m_2^2 + \mathbf{k}^2} + V, \quad (34)$$

with

$$\begin{aligned} \langle \mathbf{k} | V | \mathbf{k}' \rangle = & \frac{1}{2\sqrt{w_1 w_2 w'_1 w'_2}} \left\{ \frac{4}{3} \frac{\alpha_s}{\pi^2} \left[-\frac{1}{\mathbf{Q}^2} \left(q_{10} q_{20} + \mathbf{q}^2 - \frac{(\mathbf{Q} \cdot \mathbf{q})^2}{\mathbf{Q}^2} \right) + \right. \right. \\ & + \frac{i}{2\mathbf{Q}^2} \mathbf{k}' \times \mathbf{k} \cdot (\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2) + \frac{1}{2\mathbf{Q}^2} \left[q_{20}(\boldsymbol{\alpha}_1 \cdot \mathbf{Q}) - q_{10}(\boldsymbol{\alpha}_2 \cdot \mathbf{Q}) \right] + \\ & + \frac{1}{6} \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 + \frac{1}{4} \left(\frac{1}{3} \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 - \frac{(\mathbf{Q} \cdot \boldsymbol{\sigma}_1)(\mathbf{Q} \cdot \boldsymbol{\sigma}_2)}{\mathbf{Q}^2} \right) + \frac{1}{4\mathbf{Q}^2} (\boldsymbol{\alpha}_1 \cdot \mathbf{Q})(\boldsymbol{\alpha}_2 \cdot \mathbf{Q}) \left. \right] \\ & \left. + \frac{1}{(2\pi)^3} \int d^3 \mathbf{r} e^{i\mathbf{Q} \cdot \mathbf{r}} J^{\text{inst}}(\mathbf{r}, \mathbf{q}, q_{10}, q_{20}) \right\}, \quad (35) \end{aligned}$$

$$\begin{aligned} J^{\text{inst}}(\mathbf{r}, \mathbf{q}, q_{10}, q_{20}) = & \frac{\sigma r}{q_{10} + q_{20}} \left[q_{20}^2 \sqrt{q_{10}^2 - \mathbf{q}_T^2} + q_{10}^2 \sqrt{q_{20}^2 - \mathbf{q}_T^2} + \right. \\ & + \frac{q_{10}^2 q_{20}^2}{|\mathbf{q}_T|} \left(\arcsin \frac{|\mathbf{q}_T|}{q_{10}} + \arcsin \frac{|\mathbf{q}_T|}{q_{20}} \right) \left. \right] - \frac{\sigma}{r} \left[\frac{q_{20}}{\sqrt{q_{10}^2 - \mathbf{q}_T^2}} \left(\mathbf{r} \times \mathbf{q} \cdot \boldsymbol{\sigma}_1 + i q_{10} (\mathbf{r} \cdot \boldsymbol{\alpha}_1) \right) \right. \\ & \left. + \frac{q_{10}}{\sqrt{q_{20}^2 - \mathbf{q}_T^2}} \left(\mathbf{r} \times \mathbf{q} \cdot \boldsymbol{\sigma}_2 - i q_{20} (\mathbf{r} \cdot \boldsymbol{\alpha}_2) \right) \right]. \quad (36) \end{aligned}$$

In eq.s (34-36) \mathbf{k}' and \mathbf{k} denote the final and the initial center of mass momentum of the quark; $w_j = \sqrt{m_j^2 + \mathbf{k}^2}$, $w'_j = \sqrt{m_j^2 + \mathbf{k}'^2}$, $\mathbf{q} = \frac{\mathbf{k} + \mathbf{k}'}{2}$, $\mathbf{Q} = \mathbf{k}' - \mathbf{k}$, $q_{j0} = \frac{w_j + w'_j}{2}$; $q_T^h = (\delta^{hk} - \hat{r}^h \hat{r}^k) q^k$ is the transverse momentum, α^k are the usual Dirac matrices $\gamma^0 \gamma^k$, and $\sigma^k = 1/2 \varepsilon^{knm} \sigma^{nm}$ the 4×4 spin matrices.

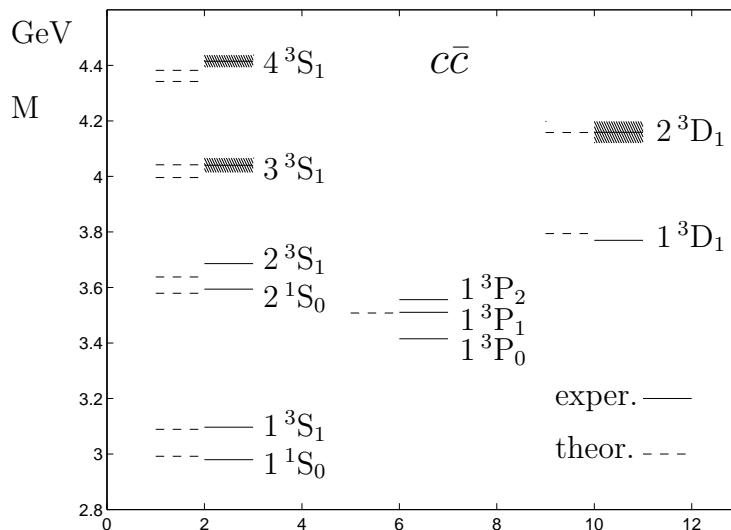


Figure 1: Charmonium spectrum.

In spite of its complication the above expression has various significant limit cases that corresponds to models successfully used in different areas. In the static limit it gives the local potential

$$V = -\frac{4}{3} \frac{\alpha_s}{r} + \sigma r. \quad (37)$$

In the heavy masses limit, by an $\frac{1}{m}$ expansion and an appropriate Foldy-Wouthuysen transformation, it reduces to the potential discussed in ref. [2], If the spin dependent terms are neglected, V becomes identical (apart from a question of ordering) to the first order expansion of the potential corresponding to the relativistic flux tube model [2, 5].

We have attempt to apply the actual V as given by (35) and (36) to the determination of the spectrum. In the preliminary calculation we have performed up to now the spin orbit terms have been omitted, due to their complication; however, the hyperfine terms have been included.

The numerical procedure we have followed consists in solving first the eigenvalue equation for the static potential (37) by the Rayleigh-Ritz method and then in evaluating the quantities $\langle H \rangle$ for the eigenfunctions obtained in the first step. We have adopted the following parameters: $\alpha_s = 0.363$, $\sigma = 0.175 \text{ GeV}^2$, $m_c = 1.405 \text{ GeV}$, $m_b = 4.81 \text{ GeV}$, $m_s = 200 \text{ MeV}$, $m_u = 10 \text{ MeV}$. The first four values have to be compared with those obtained from heavy quarkonium fits (e.g. [3, 5]) and apart from the possibility of a small rearrangement are completely determined by these. The light quark masses have been fixed a priori on typical current values as reported by the Particle Data Group.

By such parameters one succeeds to reproduce reasonably well the not only the bottonium and the charmonium spectrum, but also the Regge trajectories (with

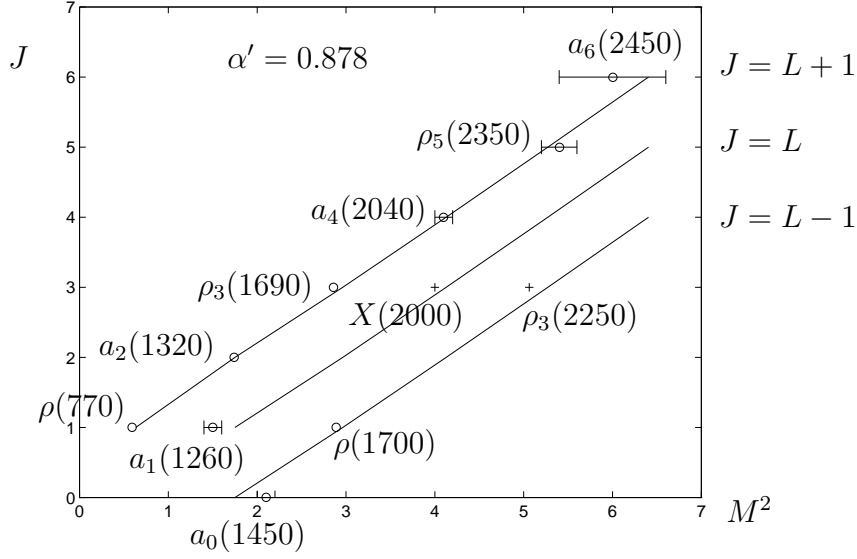


Figure 2: Ground triplet $u\bar{u}$ Regge trajectories. Theoretical results (full line) compared with experimental data (circlet). Cross denote less established masses.

Table 1: Theoretical results for $u\bar{c}$, $u\bar{b}$, $s\bar{c}$, $s\bar{b}$ systems (MeV). Experimental data are enclosed in brackets.

State	$u\bar{c}$	$u\bar{b}$	$s\bar{c}$	$s\bar{b}$
1S	1973 (1973 \pm 1)	5326 (5313 \pm 2)	2080 (2076.4 \pm 0.5)	5418 (5404.6 \pm 2.5)
2S	2600 (2623 \pm ?) ^a	5906 (5897 \pm ?) ^a	2713	6004
1P	2442 (2438 \pm ?) ^b	5777 (5825 \pm 14) ^c	2528 (2535.35 \pm 0.34)	5848 (5853 \pm 15)

^aObtained from preliminary *Delphi* data $m(D^{*'}) = 2637 \pm 8$ MeV, $m(B^{*'}) = 5906 \pm 14$ MeV [7] subtracting 1/4 theoretical hyperfine splitting reported in table 2.

^bEstimated from $m(D_2^*) = 2459 \pm 4$ MeV, $m(D_1) = 2427 \pm 5$ MeV.

^cFrom preliminary *Delphi* data [7].

Table 2: Theoretical results for $q\bar{q}$ hyperfine splitting (MeV). Experimental data are enclosed in brackets.

State	$u\bar{c}$	$u\bar{b}$	$c\bar{c}$	$b\bar{b}$	$s\bar{c}$	$s\bar{b}$
1S	111 (141 \pm 1)	59 (46 \pm 3)	97 (117 \pm 2)	102	108 (144)	60 (47 \pm 4)
2S	59	38	59 (92 \pm 5)	42	62	40

correct slope and intercepts) for the ground triplet states of the $u\bar{u}$, $u\bar{s}$, $s\bar{s}$ systems, and the known spin averaged states for the light-heavy systems. Notice that no ad hoc constant has been added to the potential and that in particular the heavy-light sector is completely parameter free.

Our results have been reported in full in ref [4], where even some details on the the numerical difficulties are explained. Here as an example for the heavy-heavy systems we report in fig. 1 the charmonium spectrum and for the light-light systems we report in fig. 2 the ρ Regge trajectory. The results for heavy-light systems are reported in table 1 and compared with the experimental spin averaged masses by using the theoretical splitting of table 2 where the singlet states are not available.

References

- [1] N. Brambilla, E. Montaldi and G.M. Prosperi, Phys. Rev. D 54 (1996) 3506; G.M. Prosperi, hep-th/9709046.
- [2] N. Brambilla and G.M. Prosperi, in *Quark Confinement and the Hadron Spectrum*, World Scientific, Singapore, I (1995), p. 113; II (1997), p. 111; W. Lucha, F. F. Schöberl and D. Gromes, Phys. Rep. 200 (1990), 127.
- [3] N. Brambilla and G.M. Prosperi, Phys. Lett. B 236 (1990) 69; A. Barchielli, N. Brambilla and G.M. Prosperi, Il Nuovo Cimento 103 A (1990) 59; F. Halzen, C. Olson, M.G. Olsson, and M.L. Stong, Phys. Rev. D 47 (1993) 3013; and references therein.
- [4] M. Baldicchi and G.M. Prosperi, Phys. Lett. B 436, 145 (1998); hep-ph/9803390.
- [5] A.Yu. Dubin, A.B. Kaidalov and Yu.A. Simonov, Phys. Lett. B 323 (1994) 41; M.G. Olsson, in *Quark Confinement and the Hadron Spectrum*, World Scientific, Singapore, (1994), p. 76 and references therein.
- [6] Particle Data Group, R.M. Barnett *et al.*, Phys. Rev. D 54 (1996).
- [7] A. Pullia (private communication).